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RESPONSE OF A SPHERICAL CAVITY IN AN ELASTIC VISCOPLASTIC MEDIUM UNDER A VARIABLE INTERNAL PRESSURE

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Abstract—A closed-form solution is given for the problem of a spherical cavity in an infinite elastoviscoplastic medium with kinematic hardening, when the cavity is subjected to an internal pressure that varies in any prescribed way.

Under certain assumptions this problem takes account of the unit weight of the medium, and is particularly applicable to the deep underground storage of natural gas in rock salt.

Some typical loading cases are shown.

NOTATION

r, θ, φ	spherical coordinates
$\sigma_{rr}, \sigma_{\theta\theta} = \sigma_{\varphi\varphi}$	radial and lateral principal stresses
$\sigma_i(t)$	radial stress applied at the wall of the cavity
u	radial displacement
ϵ_{vp}	radial viscoplastic deformation
$(\dot{\ })$	total derivative with respect to time
E, ν	elastic constants (Young modulus, Poisson's ratio)
E'	strain-hardening modulus
η	viscosity constant for viscoplasticity
S	initial uniaxial yield stress
x	radius of the region in viscoplastic loading
e	maximum value of x during its evolution
$\sigma_e(t)$	$\sigma_{rr}(e, t)$
α	$\dfrac{E'}{\eta} + \dfrac{E}{2\eta(1-\nu)}$
τ	α^{-1} (time constant)
$h(x^3)$	intermediate function. $h(x^3) = \dfrac{E'}{\eta}x^3 + \dfrac{E}{2\eta(1-\nu)}(1 + \text{Log}\, x^3)$
L, U	Indices: L = local viscoplastic loading (with respect to the frontier x); U = local viscoplastic unloading (with respect to the frontier x). Example: t_L: last time when the frontier has expanded to $r = x$; t_U: last time when the frontier has regressed to $1 = x$

1. INTRODUCTION

In quasi-static viscoplastic problems, few solutions are available in the literature which do not involve onerous numerical calculations. In the case of spherical symmetry, e.g. the problem of a spherical cavity in an infinite medium, solutions in closed form have been given by Wierzbicki[7], Aufaure[1], Tijani[6], for the special case where the pressure in the cavity was assumed to vary in a monotonic way. In this paper, a closed form solution is given for a quite general variation of the pressure and for a Bingham material with kinematic hardening. For the sake of simplicity, we suppose the medium to be infinite, but an extension of the calculation to a medium having a finite outer radius, or to the problem with cylindrical symmetry is quite easy.†

This problem is specifically appropriate to the deep underground storage of hydrocarbons in rocksalt: Laboratory investigations on rocksalt, as well as analysis of *in situ* data on such storage caverns, has led to the conclusion that the rocksalt medium behaves as a viscoplastic material[2–8]; moreover, the pressure in the cavities may vary with a high amplitude, as it does in those filled with natural gas.

†The theoretical framework of this study is given in more detail in Mandel[2] and Q. S. Nguyen [4].

2. DESCRIPTION OF THE PROBLEM UNDER STUDY

We consider the following basic configuration (Fig. 1): a spherical cavity of unit radius is located at depth h below the ground surface, and is filled with a fluid of the same unit weight γ as the medium. In a first step, the stress state inside the medium is taken to be hydrostatic:

$$\sigma_{ij}^{0} = -(\gamma h - \gamma z)\delta_{ij}.$$

This means that the pressure in the cavity is given initially by:

$$P_i = \gamma h.$$

Later on, we shall allow this pressure to vary in any prescribed way. As has been shown by Mandel[4], such a problem can be solved as a problem in a weightless medium, when substituting for σ_{ij}, the stress tensor σ^*_{ij}:

$$\sigma^*_{ij} = \sigma_{ij} - \sigma^0_{ij}.$$

In this way, the gravity term γ disappears in the equation of equilibrium although the flow law, which does not depend on the mean stress, is unaffected by the transformation.

Therefore the initial stress state is virgin, whereas the cavity is submitted to the time-dependent pressure:

$$P^*_i(t) = P_i(t) - \gamma h.$$

Since we are concerned with deep cavities (e.g. $h = 20$), the displacements at a distance h from the center of the cavity are negligible, and the stress state can be considered there also to be virgin.

We can therefore consider the equivalent problem of a large hollow sphere, free from stress on its external surface ($r = h$), and submitted to the pressure $P^*_i(t)$ on its internal surface ($r = 1$). For the sake of simplicity, the external radius is supposed here to be infinite, but γh is still taken as the geostatic pressure at the depth of the cavity.

Remark. The hypothesis of an initial hydrostatic stress state is rather reasonable for a viscous material with a small yield stress, such as rocksalt at great depth[2–8]; but the hypothesis of the equality of the densities of the fluid (natural gas.....) and the medium can be criticized. For this reason, the solution proposed here may be considered as an approach to the basic problem.

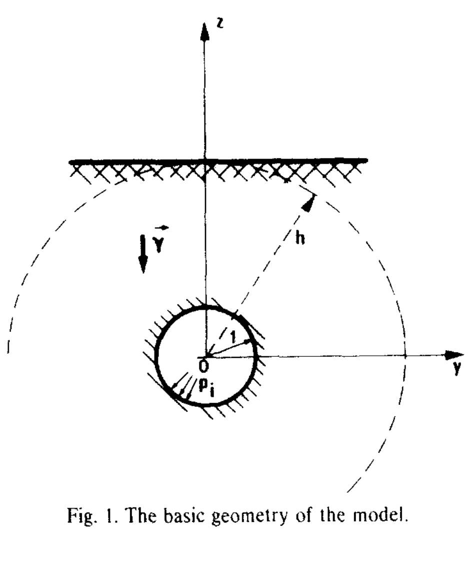

Fig. 1. The basic geometry of the model.

2.1 *Equations of the schematized problem*

Since we have a spherical symmetry, we shall use spherical coordinates (r, θ, φ), and note the principal stresses by σ_{rr} and $\sigma_{\varphi\varphi} = \sigma_{\theta\theta}$, and the radial displacement by u (Fig. 2). The different parameters being functions of radius r and time t, the equations that describe the problem can be written as follows:

- *Equations of equilibrium*:

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0. \quad (1)$$

Boundary conditions:

$$\sigma_{rr}(1, t) = \sigma_i(t), \text{with} : \sigma_i = -P_i^*.$$

- *Decomposition of the strain*:

$$\frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_{rr} - 2\nu\sigma_{\varphi\varphi}) + \epsilon_{vp} \quad (2)$$

$$\frac{u}{r} = \frac{1}{E}(1-\nu)\,\sigma_{\varphi\varphi} - \nu\sigma_{rr}) - \frac{1}{2}\epsilon_{vp} \quad (3)$$

where E, ν are the Young Modulus and Poisson's ration, and ϵ_{vp} is the radial viscoplastic strain.

Indeed, we suppose that the material exhibits an instantaneous elastic response and a viscoplasticity with kinematic strain-hardening, as schematized by the unidimensional rheological model in Fig. 3 (the hardening element E' may be taken to be zero in the calculation; in that case, we have the Bingham model). In the three-dimensional equations, the slider of the rheological model is represented by a Tresca criterion.

The viscoplastic strain ϵ_{vp} can then be written as follows; noting $\Psi = \sigma_{rr} - \sigma_{\varphi\varphi} - E'\epsilon_{vp}$, we have:

$$\left.\begin{array}{ll} \text{a)} & \dfrac{\partial \epsilon_{vp}}{\partial t} = 0 \text{ if } -S \leqslant \Psi \leqslant S; \\ \text{b)} & \dfrac{\partial \epsilon_{vp}}{\partial t} = \dfrac{1}{\eta}(\Psi - \omega S) \text{ otherwise, where } \omega = \operatorname{sgn}\Psi \end{array}\right\} \quad (4)$$

S is the initial yield limit; E' the strain hardening modulus; and η the coefficient of viscosity.

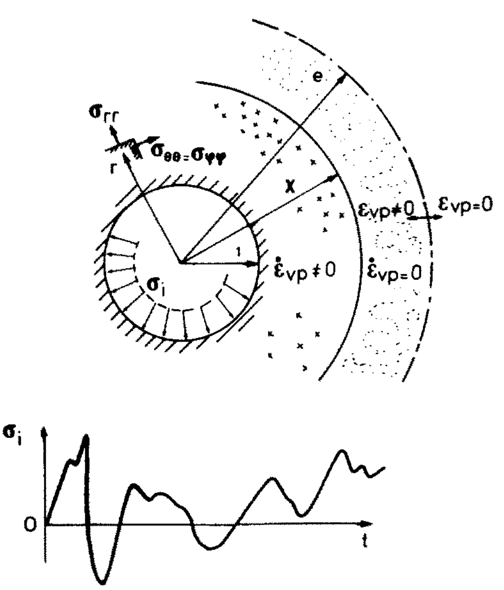

Fig. 2. Geometry of the equivalent model and notations.

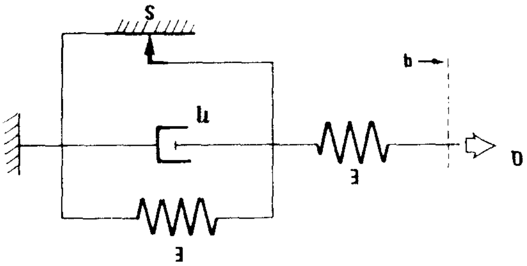

Fig. 3. Rheological model.

2.2 *Notations*

Once the yield criterion has been exceeded, a viscoplastic region ($\partial\epsilon_{vp}/\partial t \neq 0$) will develop from the cavity inside the medium. Later on, this region may regress and eventually disappear, leaving residual strains in the volumes it has reached.

In the general case, three regions can be distinguished in the medium (Fig. 2).

If we note by x the outer border of the viscoplastic region, and by e the outer border of those regions which have undergone viscoplastic strains ($e = \max\{x\}, \forall t < t(x)$), we have:

Region 1. $1 \leq r \leq x \quad \dfrac{\partial\epsilon_{vp}}{\partial t} \neq 0 \quad \epsilon_{vp} \neq 0.$

Region 2. $x \leq r \leq e \quad \dfrac{\partial\epsilon_{vp}}{\partial t} = 0 \quad \epsilon_{vp} \neq 0.$

Region 3. $e \leq r \leq \infty \quad \dfrac{\partial\epsilon_{vp}}{\partial t} = 0 \quad \epsilon_{vp} = 0.$

We note by $\sigma_e = \sigma_{rr}(e, t)$ the radial stress on the frontier: $r = e$.

2.3 *Methods*

The solution is obtained by determining the boundary $x(t)$, from which the unknown variables are then easily calculated. An example of the variation of x with time is given in Fig. 4; various cases can be distinguished:

—when the viscoplastic loading region develops in a virgin medium, i.e. for the first loading ($x = e$), the unknown x is the solution of an ordinary first order differential equation; this holds for *AB*, *DE*, *KL*.

—Otherwise, in the general case, we shall prove that:

$$H(x^3)\frac{dx^3}{dt} = F(x^3) + G(t),$$

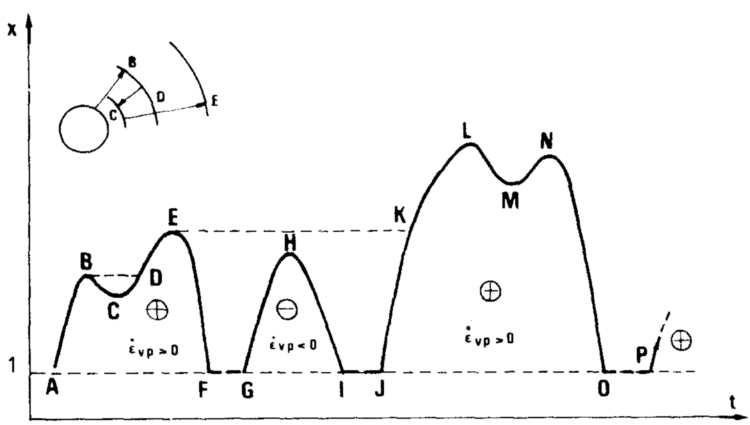

Fig. 4. Evolution of the viscoplastic loading region.

where $H(x^3)$ and $F(x^3)$ integrate the history of loading, as functions of the successive moments $t^1(x)$, $t^2(x) \ldots t^n(x)$ when the viscoplastic loading border overtake point x; this holds for BC, $EF, \ldots (\dot{x} < 0)$ or CD, $GH, \ldots (\dot{x} > 0$ and $x < e)$, Fig. 4).

3. INTRODUCTION OF AN INTERMEDIATE VARIABLE $e^3\sigma_e$

The elimination of $\sigma_{\varphi\varphi}$ and ϵ_{vp} from (1) to (3) leads to an equation between u and σ_{rr} which we can integrate from r to ∞:

$$\frac{u}{r} = \frac{1-2\nu}{E}\sigma_{rr} + \frac{C(t)}{r^3}. \tag{5}$$

$C(t)$ is a variable of integration which can be expressed as a function of $\sigma_e = \sigma_{rr}(e, t)$, by taking into account that u must be continuous through $r = e$. Since we have, by the elastic solution:

$$\frac{u}{r}(r = e) = -\frac{(1+\nu)}{2E}\sigma_e,$$

eqn (5) yields:

$$\frac{u}{r} = \frac{1-2\nu}{E}\sigma_{rr} - \frac{3}{2}\frac{(1-\nu)}{E}e^3\sigma_e. \tag{6}$$

From (6) and (3), the radial viscoplastic deformation can be written in a similar way, as a function of σ_{rr} and the unknown quantity $e^3\sigma_e$:

$$\epsilon_{vp} = \frac{1-\nu}{E}\left(r\frac{\partial\sigma_{rr}}{\partial r} + \frac{3e^3\sigma_e}{r^3}\right). \tag{7}$$

Note that (7) is true every where in the medium (the flow law (4) has not been used yet). In particular, we can check that $\epsilon_{vp} \equiv 0$ in zone 3, which is elastic.

If there is a zone which undergoes viscoplastic loading (zone 1) we also have , for $r \in [1, x]$, eqn (4) which can be written as follows:

$$\frac{\partial\epsilon_{vp}}{\partial t} = \frac{1}{\eta}\left[-\frac{r}{2}\frac{\partial\sigma_{rr}}{\partial r} - E'\epsilon_{vp} - \omega S\right]. \tag{8}$$

One or the other of the functions ϵ_{vp}, σ_{rr} can now be eliminated from eqns (7) and (8) to obtain an equation involving the stress σ_{rr} or one involving the strain ϵ_{vp}. Taking into account the boundary conditions or $r = 1$ and $r = x$ then yields an equation for the unknown $(e^3\sigma_e)$.

Remark on the variation of $e^3\sigma_3$

We observe that:

—either e is a function of time; in this case, $x = e$, and the points at the border e are submitted to a first viscoplastic loading, so that eqn (4) yields:

$$(\sigma_{rr} - \sigma_{\varphi\varphi})_{(r=e)} = \omega S.$$

Then, from the elastic solution we obtain that σ_e remains constant:

$$\sigma_e = \sigma_{rr}(e, t) = 2\omega S.$$

—Or e is constant; in that case, $x < e$ and σ_e is a function of time. These considerations show us that $\frac{d}{dt}(e^3\sigma_e)$ will mean either $e^3\frac{d}{dt}\sigma_e$ (e constant) or $\sigma_e\frac{d}{dt}e^3$ (σ_e constant).

4. RELATIONS OBTAINED ON $r = x$ (REGION 1)

4.1 *The equation involving stresses*

We eliminate ϵ_{vp} between eqns (7) and (8), thus obtaining an equation for the stress $\sigma_{rr}(r, t)$:

$$\frac{\partial}{\partial t}\left[\frac{\partial \sigma_{rr}}{\partial r} + \frac{3e^3\sigma_e}{r^4}\right] = -\alpha\frac{\partial \sigma_{rr}}{\partial r} - \frac{3E'}{\eta}\frac{e^3\sigma_e}{r^4} - \frac{\omega ES}{\eta(1-\nu)}\frac{1}{r} \tag{9}$$

where

$$\alpha = \frac{E'}{\eta} + \frac{E}{2\eta(1-\nu)}.$$

Integrating (9) for fixed t, between $r = 1$ and $r = x$, we have:

$$\left.\begin{aligned}\frac{\partial}{\partial t}[\sigma_{rr}(x,t) - \sigma_{rr}(1,t)] - \left(\frac{1}{x^3} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e) &= -\alpha[\sigma_{rr}(x,t) - \sigma_{rr}(1,t)] \\ &\quad + \frac{E'}{\eta}\left(\frac{1}{x^3} - 1\right)e^3\sigma_e - \frac{\omega ES}{3\eta(1-\nu)}\,\mathrm{Log}\,x^3.\end{aligned}\right\} \tag{10}$$

Notice that $\sigma_{rr}(1,t) = \sigma_i(t)$ and that $(\partial/\partial t)[\sigma_{rr}(1,t)] = \dot{\sigma}_i(t)$ (the dot means total derivative with respect to time). Moreover, we shall prove that (see Appendix 1):

$$\frac{\partial}{\partial t}[\sigma_{rr}(x,t)] = \frac{1}{x^3}\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e); \tag{11}$$

hence, eqn (10) reduces finally to:

$$\dot{\sigma}_i + \alpha\sigma_i = \frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e) + \alpha\sigma_{rr}(x,t) + \frac{E'}{\eta}\left(1 - \frac{1}{x^3}\right)e^3\sigma_e + \frac{\omega ES}{3\eta(1-\nu)}\,\mathrm{Log}\,x^3. \tag{12}$$

We now distinguish between two cases:

(*a*) *Case when* e *varies with time* (e = x *and* σ_e *is constant*). The points at the border e are submitted to a first viscoplastic loading. Equation (12) reduces to the formula obtained by Tijani[6]: e^3 is a solution of the first order differential equation:

$$\dot{\sigma} + \alpha\sigma_i = \omega\frac{2S}{3}\left[\frac{\mathrm{d}}{\mathrm{d}t}e^3 + \frac{E'}{\eta}e^3 + \frac{E}{2\eta(1-\nu)}(1 + \mathrm{Log}\,e^3)\right]. \tag{13}$$

(*b*) *The general case.* Now, consider $e \neq x$.

First, note that $(\mathrm{d}/\mathrm{d}t)\,\sigma_{rr}(x,t)$ can be expressed as a function of $e^3\sigma_e$ and x, since we have:

$$\frac{\partial}{\partial t}\sigma_{rr}(x,t) = \frac{1}{x^3}\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e) \text{ (see (11)).}$$

- The left side of (9) vanishes for $r = x$, since $(\partial\epsilon_{vp}/\partial t)(x,t) = 0$; this yields:

$$\frac{\partial}{\partial r}\sigma_{rr}(x,t) = -\frac{1}{\alpha}\left[\frac{3E'}{\eta}\frac{e^3\sigma_e}{x^3} + \frac{\omega ES}{\eta(1-\nu)}\frac{1}{x}\right].$$

Let us now differentiate (12) with respect to time; according to the remark above, the quantity $(\mathrm{d}/\mathrm{d}t)\sigma_{rr}(x,t)$ can be eliminated, and we obtain:

$$\ddot{\sigma}_i + \alpha\dot{\sigma}_i = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^3\sigma_e) + \alpha\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e) + \frac{E}{2\eta(1-\nu)}\left(\frac{1}{x^3} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e). \tag{14}$$

We have thus obtained a differential equation between x^3 and $e^3\sigma_e$ (notice that $\dot{x}$ has disappeared from this equation); another relation is needed between these two variables, which will be obtained in a later paragraph. The reader will remark that (14) includes the derivative of (13), when setting $x = e$.

4.2 *The equations involving the strain* ϵ_{vp}

Let us now eliminate σ_{rr} between (7) and (8), obtaining an equation for ϵ_{vp}:

$$\frac{\partial \epsilon_{vp}}{\partial t} + \alpha \epsilon_{vp} = -\frac{\omega S}{\eta} + \frac{3}{2\eta} \frac{e^3 \sigma_e}{r^3} \tag{15}$$

(*a*) *Viscoplastic local loading followed by local unloading.* Let t_L (load) and t_U (unload) be two moments defined by: (Fig. 5) (the loading is not necessarily a first loading)

$$x(t_L) = x(t_U)$$

$$\dot{x}(t_L) > 0, \quad \dot{x}(t_U) < 0$$

$$x(t) \neq x(t_L) \text{ for } t_L < t < T_U.$$

Then we have:

$$\frac{\partial \epsilon_{vp}}{\partial t}(x(t_L), t_L) = 0$$

$$\frac{\partial \epsilon_{vp}}{\partial t}(x(t_U), t_U) = 0.$$

We can then differentiate (15) with respect to time keeping $r = x(t_L) = x(t_U)$ fixed:

$$t_L \leq t \leq t_U: \quad \frac{\partial^2 \epsilon_{vp}(x(t_L), t)}{\partial t^2} + \alpha \frac{\partial \epsilon_{vp}(x(t_L), t)}{\partial t} = \frac{3}{2\eta x^3(t_L)} \frac{\mathrm{d}}{\mathrm{d}t}(e^3 \sigma_e),$$

this can be multiplied by $\exp(\alpha t)$ and then integrated between t_L and t_U:

$$\begin{cases} 0 = \displaystyle\int_{t_L}^{t_U} \exp(\alpha t) \frac{\mathrm{d}}{\mathrm{d}t}(e^3 \sigma_e)\, \mathrm{d}t \\ x(t_U) = x(t_L). \end{cases} \tag{16}$$

Considering this integral as a function of the upper limit t_U, we differentiate it:

$$0 = \exp(\alpha t_U) \widehat{e^3 \dot{\sigma}_e}(t_U) - \exp(\alpha t_L) \widehat{e^3 \dot{\sigma}_e}(t_L) \frac{\mathrm{d}t_L}{\mathrm{d}t_U}$$

Hence:

$$\frac{\exp(\alpha t_U) \widehat{e^3 \dot{\sigma}_e}(t_U)}{\widehat{\dot{x}^3}(t_U)} = \frac{\exp(\alpha t_L) \widehat{e_3 \dot{\sigma}_e})t_L)}{\widehat{\dot{x}^3}(t_L)}. \tag{17}$$

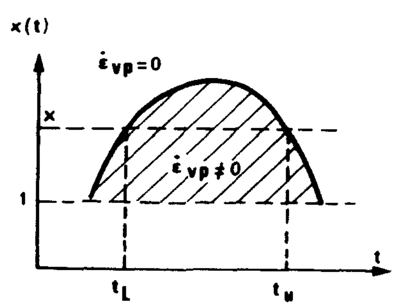

Fig. 5. Increase and decrease of the elastoviscoplastic border.

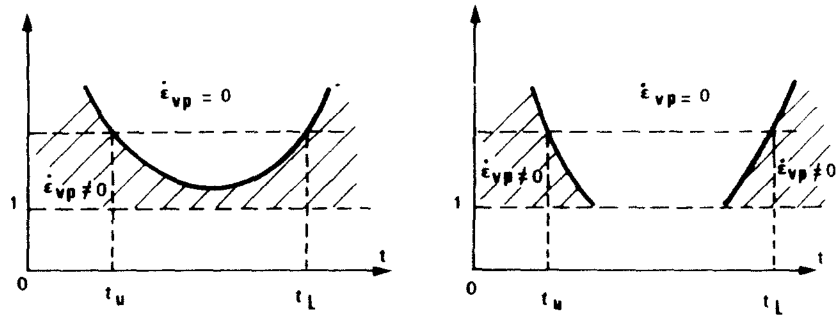

Fig. 6. Decrease and increase of the elastoviscoplastic border (a) the viscoplastic zone does not vanish between t_U and t_L, (b) the viscoplastic zone vanishes between t_U and t_L.

(*b*) *Viscoplastic local unloading followed by local loading.* Let t_U and t_L be two moments defined by (Fig. 6):

$$t < t_U \quad \frac{\partial \epsilon_{vp}}{\partial t}(t) \neq 0 \qquad \operatorname{sgn} \frac{\partial \epsilon_{vp}}{\partial t}(t) = \omega_U$$

$$t > t_L \quad \frac{\partial \epsilon_{vp}}{\partial t}(t) \neq 0 \qquad \operatorname{sgn} \frac{\partial \epsilon_{vp}}{\partial t}(t) = \omega_L$$

$$t_U < t < t_L \quad \frac{\partial \epsilon_{vp}}{\partial t}(t) = 0 \text{ or:} \qquad \epsilon_{vp}(t) = \epsilon_{vp}(t_U) = \epsilon_{vp}(t_L).$$

Note that in Fig. 6(a), $\omega_U = \omega_L$; but ω_L may be equal to $\pm\,\omega_U$ in Fig. 6(b), (see also Fig. 4). Equation (15) then gives:

$$e^3\sigma_e(t_U) - e^3\sigma_e(t_L) = \frac{2S}{E}(\omega_U - \omega_L)x^3(t_L) = \frac{2S}{E}(\omega_U - \omega_L)x^3(t_U). \tag{18}$$

Differentiating this expression with respect to x^3 (or with respect to t_U as in (17)), we obtain an expression similar to (17):

$$\frac{\widehat{e^3\sigma_e(t_U)}}{\dot{\widehat{x^3}}(t_U)} - \frac{\widehat{e^3\sigma_e(t_L)}}{\dot{\widehat{x^3}}(t_L)} = \frac{2S}{3}(\omega_U - \omega_L) \tag{19}$$

5. EVOLUTION OF THE VISCOPLASTIC LOADING ZONE IN THE GENERAL CASE

Let us return to eqn (14) and take into account the results obtained in 4.2.

5.1 *Viscoplastic local loading followed by local unloading*

$$t_L < t < t_U \text{ (Fig. 4).}$$

We multiply (14) on each side by $\exp \alpha t$, and integrate it between t_L and t_U =

$$\exp(\alpha t_U)[\dot{\sigma}_i(t_U) - \dot{\widehat{e^3\sigma_e(t_U)}}] - \exp(\alpha t_L)[\dot{\sigma}_i(t_L) - \dot{\widehat{e^3\sigma_e(t_L)}}] =$$

$$\frac{E}{2\eta(1-\nu)}\int_{t_L}^{t_U} \exp(\alpha t)\left(\frac{1}{x^3} - 1\right)\dot{\widehat{e^3\sigma_3}}\,dt.$$

The right side of this equation is zero, due to (17). Indeed, if we suppose x_{max} to be the only maximum of x between t_L and t_U, we deduce from (17), by changing the integration variable, that any integral of the following form vanishes:

$$\int_{t_L}^{t_U} \phi(x^3(t)) \exp(\alpha t) \dot{\widehat{e^3\sigma_e}}\, dt = \int_{t_L}^{t_U} \phi(x^3(t)) \exp(\alpha t) \frac{\dot{x}^3 \dot{\widehat{e^3\sigma_e}}}{\dot{x}^3}\, dt$$

$$= \int_{x^3(t_L)=x^3(t_U)}^{x^3_{max}} \phi(x^3)\left[\exp(\alpha t_U)\frac{\dot{\widehat{e^3\sigma_e}}(t_U)}{\dot{x}^3(t_U)} - \exp(\alpha t_L)\frac{\dot{\widehat{e^3\sigma_e}}(t_L)}{\dot{x}^3(t_L)}\right] dx^3$$

The same proof can easily be extended to the case where there exist several extrema for x between t_L and T_u.

Hence:

$$\exp(\alpha t_U)(\dot{\sigma}_i(t_U) - \dot{\widehat{e^3\sigma_e}}(t_U)) = \exp(\alpha t_L)(\dot{\sigma}_i(t_L) - \dot{\widehat{e^3\sigma_e}}(t_L)). \qquad (20)$$

When combining (20) with (17) to eliminate $e^3\sigma_e(t_U)$ we obtain $\dot{x}^3(t_U)$ finally as a function of t_U and of quantities which have already been calculated at the moment t_L:

$$\dot{x}^3(t_U) = \dot{x}^3(t_L)\left\{1 + \frac{\dot{\sigma}(t_U)\exp\alpha(t_U - t_L) - \dot{\sigma}_i(t_L)}{\dot{\widehat{e^3\sigma_e}}(t_L)}.\right\} \qquad (21)$$

5.2 *Viscoplastic local unloading followed by local loading*

$$t_U < t < t_L \quad \text{(Fig. 5)}.$$

Proceeding in an analogous way as for the calculation of (20), we have:

$$\int_{t_U}^{t_L}\left(\frac{1}{x^3(t)} - 1\right)\dot{\widehat{e^3\sigma_e}}\, dt = \int_{x^3(t_U)=x^3(t_L)}^{x^3{}_{min}} (\omega_U - \omega_L)\left(\frac{1}{x^3} - 1\right)\frac{2S}{3}\, dx^3$$

$$= \frac{2S}{3}(\omega_U - \omega_L)[\text{Log}\, x^3 - x^3]_{x^3(t_U)=x^3(t_L)}^{x^3_{min}} \qquad (22)$$

(We suppose once more, to simplify the writing, that x^3 has but one minimum between t_U and t_L.)

We can then distinguish two cases:

—x min $\neq 1$ hence $\omega_U = \omega_L$, and the integral is zero (Fig. 6a);
—x^3 min $= 1$ and it may be the case that $\omega_U = -\omega_L$ (Fig. 6b).

We can therefore write formally x^3 min $= 1$ in eqn (22); if we note:

$$h(x^3) = \frac{E'}{\eta}x^3 + \frac{E}{2\eta(1-\nu)}(1 + \text{Log}\, x^3),$$

then the integration of (14) between t_U and t_L yields finally:

$$\alpha\sigma_i(t_U) + \dot{\sigma}_i(t_U) - \dot{\widehat{e^3\sigma_e}}(t_U) - \omega_U h(x^3) = \alpha\sigma_i(t_L) + \dot{\sigma}_i(t_L) - \dot{\widehat{e^3\sigma_e}}(t_L) - \omega_L h(x^3). \qquad (23)$$

As in the first case, we can eliminate $(e^3\sigma_e(t_L))$ between (23) and (19) to get $\dot{x}^3(t_L)$ as a function of t_L and of known quantities already calculated at the moment t_U:

$$\left.\begin{aligned}&\frac{\widehat{e^3\sigma_e(t_U)} + \alpha\sigma_i(t_L) + \dot{\sigma}_i(t_L) - \alpha\sigma_i(t_U) - \dot{\sigma}_i(t_U) + \frac{2S}{E}(\omega_U - \omega_L)h(x^3)}{\widehat{\dot{x}^3(t_L)}}\\ &= \frac{\widehat{e^3\sigma_e(t_U)}}{\widehat{\dot{x}^3(t_U)}} + \frac{2S}{E}(\omega_L - \omega_U).\end{aligned}\right\} \quad (24)$$

The reader will notice that (23) reduces to the first loading case (formula (13)) if we set the left side equal to zero.

6. CONCLUSION

6.1 *Numerical procedure*

The preceeding results allow us to calculate the variation of x^3 with time as the solution of a first order differential equation. Referring to Fig. 7.

(a) *Between* A *and* B, *eqn* (13) *can be used.*

$$\frac{dx^3}{dt} = -\frac{E'}{\eta}x^3 - \frac{E}{2\eta(1-\nu)}(1 + \text{Log}\,x^3) + \frac{\dot{\sigma}_i + \alpha\sigma_i}{\omega\frac{2S}{3}} \quad (13)'$$

since $\sigma_i(t)$ is a given function; the solution of this equation, together with the relevant initial conditions, can be obtained as $x_L{}^3 = f_L(t)$ or $t_L = f_L^{-1}(x^3)$.

(*b*) *Between* B *and* C, *eqn* (21) *can be used.*

$$\frac{dx^3}{dt} = \hat{\dot{x}}^3(t_L)\left\{1 + \frac{\dot{\sigma}_i(t)\exp\alpha(t - t_L) - \dot{\sigma}_i(t_L)}{\widehat{e^3\dot{\sigma}_e(t_L)}}\right\}. \quad (21)'$$

The solution $t_L = f^{-1}(x)$ calculated in (a) above must be used here; then this eqn (21)', may be solved to obtain $x_U{}^3 = g_U(t)$ or $t_U = g_U^{-1}(x^3)$.

(*c*) *Between* C *and* D, *eqn* (24) *can be used.*

$$\frac{dx^3}{dt} = \frac{\widehat{e^3\sigma_e(t_U)} + \alpha\sigma_i(t) + \dot{\sigma}_i(t) - \alpha\sigma_i(t_U) - \dot{\sigma}(t_U) + \frac{2S}{3}(\omega_U - \omega_L)\,h(x^3)}{\frac{\widehat{e^3\dot{\sigma}_e(t_U)}}{\widehat{x^3(t_U)}} + \frac{2S}{3}(\omega_L - \omega_U)} \quad (24)'$$

Once more, the solution $t_U = g_U{}^{-1}(x^3)$ calculated above must be used here, etc...

When substituting $t_L = f_L{}^{-1}(x^3)$ into (21)' or $t_U = g_U{}^{-1}(x^3)$ into (24)' we obtain the form

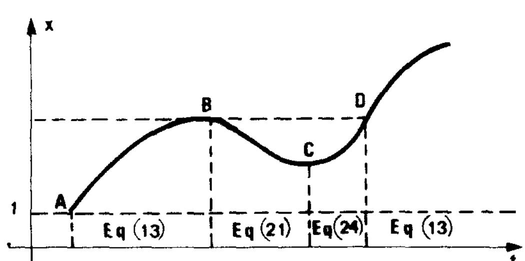

Fig. 7. Summary of different solutions for the evolution of $x^3(t)$.

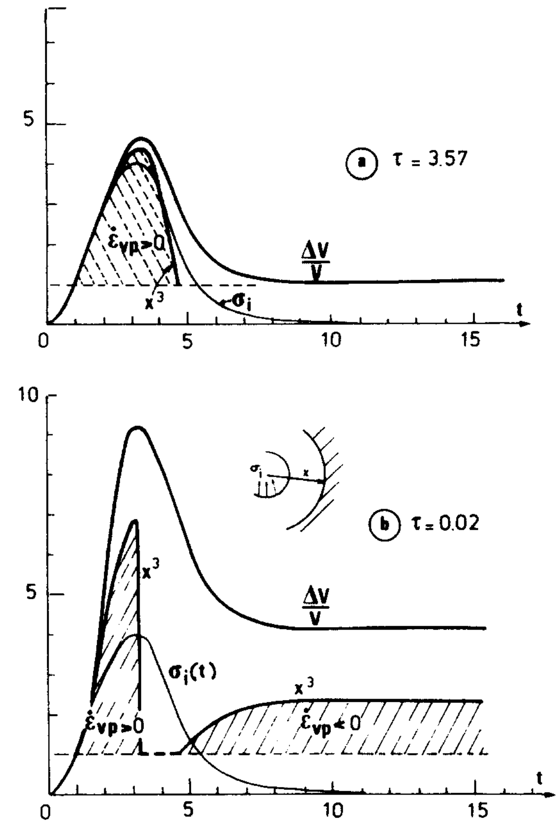

Fig. 8. "Loading during a finite interval of time", (a) high viscosity, (b) low viscosity.

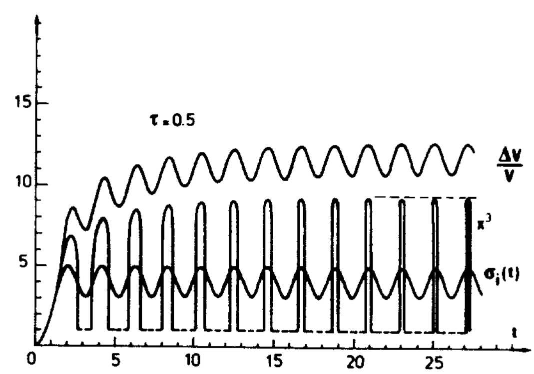

Fig. 9. "Cyclic loading".

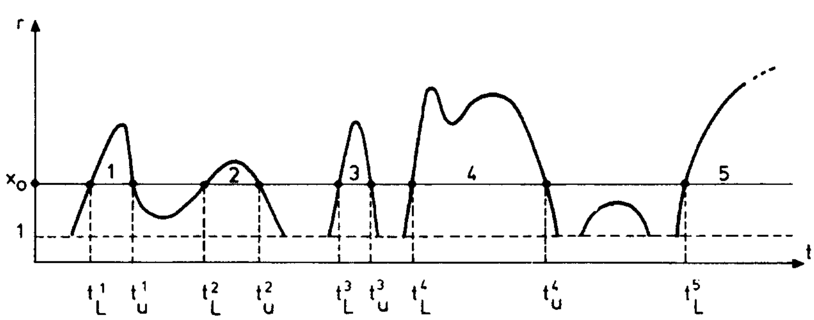

Fig. 10. "Evolution of the viscoplastic frontier vs time".

mentioned in 2.3:

$$\frac{dx^3}{dt} = \frac{F(x^3) + G(t)}{H(x^3)}.$$

It may be expressed explicitly as a function of successive moments t_L and t_U when the frontier of the viscoplastic zone has overtaken point x (see the recurrence formulas in Appendix 2).

If the time dependence of the stress $\sigma_i(t)$ applied at the wall of the cavity is not simple, the solutions of eqns (13)′, (21)′, (24)′ must be in general calculated on a small computer. At each step, the calculations of $[t_L(x^3)$ or $t_U(x^3)]$ must be stored.

6.2 *Numerical examples*

The following variables have been plotted on one diagram as functions of time: the function $X(t)$ standing for the evolution of the viscoplastic frontier, the associated volumic variation $(\Delta V/V)(t)$, and the prescribed stress $\sigma_i(t)$.

The three functions are plotted in a dimensionless form, by taking their respective values to be equal to unity when viscoplasticity appears in the medium for the first time.

Thus the three curves pass through the same initial point; moreover, if we allow the viscosity constant to approach infinity, it can be shown that these curves become identical. This means that the behaviour of the medium is then elastic.

Two kinds of applied stress $\sigma_i(t)$ have been tested:

● *Figure* 8. *$\sigma_i(t)$ increases to a maximum and then decreases to zero, in finite interval of time*

We can observe two different responses of the structure depending on the relative viscosity of the material, or more precisely, on the value of the time constant τ:

$$\tau = \alpha^{-1} = 2\eta(1-\nu)/(E + 2(1-\nu)E').$$

A typical plastic-like response for a rather small value of τ is seen in Fig. 8(b).

● *Figure* 9. *Cyclic variation of $\sigma_i(t)$*

The evolution of the viscoplastic zones suggests an elastic shakedown of the structure, after a great number of cycles.

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APPENDIX 1

Proof of the formula (11)

$$\frac{\partial}{\partial t}\sigma_{rr}(x,t) = \frac{1}{x^3}\frac{d}{dt}(e^3\sigma_e).$$

Two cases must be distinguished:
$a - x = e$, then σ_e is a constant (see 2.2), hence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_e = 0 = \frac{\partial}{\partial t}\sigma_{rr}(e, t) + \dot{e}\frac{\partial}{\partial r}\sigma_{rr}(e, t). \tag{A1}$$

But $\epsilon_{vp}(e, t) = 0$ and $\frac{\partial}{\partial t}\epsilon_{vp}(e, t) = 0$.

Thus, from (8), we have:

$$\frac{\partial}{\partial r}\sigma_{rr}(e, t) = -\frac{2\omega S}{e}$$

and finally, from (A1):

$$\frac{\partial}{\partial t}\sigma_{rr}(e, t) = 2\omega S\frac{\dot{e}}{e} = \frac{1}{e^3}\frac{\mathrm{d}}{\mathrm{d}t}(e^3\sigma_e).$$

$b - x \neq e$, then e is a constant (see 2.2).

We will use the letters E (elastic) and VP (viscoplastic) to distinguish the two regions delimited by the viscoplastic frontier $x = x(t)$. The different functions (σ_{rr}, u, ϵ_{vp}, etc...) are separately defined in each of the two regions: when necessary we will distinguish $\sigma_{rr}^E(r, t)$ and $\sigma_{rr}^{VP}(r, t)$, etc. In the elastic region, the incremental behavior is of an elastic type, so that, we have:

$$\frac{\partial}{\partial t}\sigma_{rr}^E(r, t) = \frac{e^3}{r^3}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_e(t). \tag{A2}$$

An identical relation will hold in the viscoplastic region, if we show that $(\partial/\partial r)\sigma_{rr}(r, t)$ is continuous across the viscoplastic border x.

First, remark that $\sigma_{rr}(r, t)$ is continuous through x, and so is its total derivative with respect to time; thus:

$$\frac{\partial}{\partial t}\sigma_{rr}^E(x, t) + \dot{x}\frac{\partial}{\partial r}\sigma_{rr}^E(x, t) = \frac{\partial}{\partial t}\sigma_{rr}^{VP}(x, t) + \dot{x}\frac{\partial}{\partial r}\sigma_{rr}^{VP}(x, t). \tag{A3}$$

Moreover, the continuity of ϵ_{vp} through x, yields from (7):

$$\frac{\partial}{\partial r}\sigma_{rr}^E(x, t) = \frac{\partial}{\partial r}\sigma_{rr}^{VP}(x, t).$$

combining this with (A3), we deduce that $(\partial/\partial r)\sigma_{rr}(x, t)$ is continuous through x; the formula (11) is then proved for the viscoplastic side of $r = x$.

APPENDIX 2

Recurrence formulas

We consider a fixed point x_0 which is subjected to several viscoplastic local loadings (see Fig. 10). $t_L{}^k$ and $t_U{}^k$ are th moments when the k^{th} viscoplastic local loading begins and ends:

$$\frac{\partial\epsilon_{vp}}{\partial t}(x_0, t_L{}^k) = \frac{\partial\epsilon_{vp}}{\partial t}(x_0, t_U{}^k) = 0$$

$$t_L{}^k < t < t_U{}^k,\ \frac{\partial\epsilon_{vp}}{\partial t}(x_0, t) \neq 0$$

$$t_U{}^k \leq t \leq t_L{}^{k+1},\ \frac{\partial\epsilon_{vp}}{\partial t}(x_0, t) = 0.$$

Or, put in another way, if $x = x(t)$ is the viscoplastic frontier:

$$x_0 = x(t_L{}^k);\quad \dot{x}(t_L{}^k) > 0$$

$$x_0 = x(t_U{}^k);\quad \dot{x}(t_U{}^k) < 0.$$

Then equations (17) and (19) can be rewritten as follows:

$$\frac{\exp(\alpha t_U{}^n)\widehat{e^3\dot{\sigma}_e(t_U{}^n)}}{\widehat{\dot{x}^3}(t_U{}^n)} = \frac{\exp(\alpha t_L{}^n)\widehat{e^3\dot{\sigma}_e(t_L{}^n)}}{\widehat{\dot{x}^3}(t_L{}^n)} \tag{17'}$$

$$\frac{\widehat{e^3\dot{\sigma}_e(t_L)}}{\widehat{\dot{x}^3}(t_L{}^n)} = \frac{\widehat{e^3\dot{\sigma}_e(t_U{}^{n-1})}}{\widehat{\dot{x}^3}(t_U{}^{n-1})} + \frac{2S}{3}(\omega_n - \omega_{n-1}). \tag{19'}$$

Let θ_n be:

$$\theta_n = t_U{}^{n-1} - t_L{}^{n-1} + t_U{}^{n-2} - t_L{}^{n-2} + \ldots - t_L{}^1.$$

From (17)′ and (19)′ we easily deduce:

$$\frac{\widehat{e^3\sigma_e(t_L{}^n)}}{\hat{x}^3(t_L{}^n)} = \frac{2S}{3}\sum_{j=1}^{n}(\omega_j - \omega_{j-1})\exp(\alpha\theta_j - \alpha\theta_n) \tag{A.4}$$

$$\frac{\widehat{e^3\sigma_e(t_U{}^n)}}{\hat{x}^3(t_U{}^n)} = \frac{2S}{3}\sum_{j=1}^{n}(\omega_j - \omega_{j-1})\exp(\alpha\theta_j - \alpha\theta_{n+1}). \tag{A.5}$$

By convention, $\omega_0 = 0$, $\theta_1 = 0$.

In a similar way, eqns (20) and (23) can be written as follows:

$$\dot{\sigma}_i(t_L{}^n) - \widehat{e^3\dot{\sigma}_e}(t_L{}^n) = \exp(\alpha t_U{}^n - \alpha t_L{}^n)[\dot{\sigma}_i(t_U{}^n) - \widehat{e^3\dot{\sigma}_e}(t_U{}^n)] \tag{20'}$$

$$\alpha\sigma_i(t_U{}^n) + \dot{\sigma}_i(t_U{}^n) - \widehat{e^3\dot{\sigma}_e}(t_U{}^n) - \omega_n h(x^3) = \alpha\sigma_i(t^{n+1}) + \dot{\sigma}_i(t^{n+1}) - \widehat{e^3\dot{\sigma}_e}(t^{n+1}) - \omega_{n+1}h(x^3). \tag{23'}$$

These two equations (20)′ and (23)′ stand for the successive viscoplastic local loadings:

$$0 = \alpha\sigma_i(t_L{}^1) + \dot{\sigma}_i(t_L{}^1) - \widehat{e^3\dot{\sigma}_e}(t_L{}^1) - \omega_1 h(x^3) \qquad L_1$$

$$\dot{\sigma}_i(t_L{}^1) - \widehat{e^3\dot{\sigma}_e}(t_L{}^1) = \exp(\alpha t_U{}^1 - \alpha t_L{}^1)[\dot{\sigma}_1(t_U{}^1) - \widehat{e^3\dot{\sigma}_e}(t_U{}^1)] \qquad U_1$$

$$\alpha\sigma_i(t_U{}^1) + \dot{\sigma}_i(t_U{}^1) - \widehat{e^3\dot{\sigma}_e}(t_U{}^1) - \omega_1 h(x^3) = \alpha\sigma_i(t_L{}^2) + \dot{\sigma}_i(t_L{}^2) - \widehat{e^3\dot{\sigma}_e}(t_L{}^2) - \omega_2 h(x^3) \qquad L_2$$

$$\vdots$$

$$\dot{\sigma}_i(t_L{}^{n-1}) - \widehat{e^3\dot{\sigma}_e}(t_L{}^{n-1}) = \exp(\alpha t_U{}^{n-1} - \alpha t_L{}^{n-1})[\dot{\sigma}_i(t_U{}^{n-1}) - \widehat{e^3\dot{\sigma}_e}(t_U{}^{n-1})] \qquad U_{n-1}$$

$$\alpha\sigma_i(t_U{}^{n-1}) + \dot{\sigma}_i(t_U{}^{n+1}) - \widehat{e^3\dot{\sigma}_e}(t_U{}^{n-1}) - \omega_{n-1}h(x^3) = \alpha\sigma_i(t_L{}^n) + \dot{\sigma}_i(t_L{}^n) - \widehat{e^3\dot{\sigma}_e}(t_L) - \omega_n h(x^3) \qquad L_n$$

$$\dot{\sigma}_i(t_L{}^n) - \widehat{e^3\dot{\sigma}_e}(t_L{}^n) = \exp(\alpha t_U{}^n - \alpha t_L{}^n)[\dot{\sigma}_i(t_U{}^n) - \widehat{e^3\dot{\sigma}_e}(t_U{}^n)] \qquad U_n$$

In order to eliminate $\dot{\sigma}_i - \widehat{e^3\dot{\sigma}_e}$, we multiply lines L_k and U_k by $\exp(\alpha\theta_u)$ and sum the lines L_1 to L_n:

$$\sum_{k=1}^{n}\exp(\alpha\theta_k)[\alpha\sigma_i(t^{k-1}) - \omega_{k-1}h(x^3)] = \exp(\alpha\theta_u)[\dot{\sigma}_i(t_L{}^n) - \widehat{e^3\dot{\sigma}_e}(t_L{}^n)] + \sum_{k=1}^{n}\exp(\alpha\theta_k)[\alpha\sigma_i(t_L{}^k) - \omega_k h(x^3).$$

From (A4) we have:

$$\exp(\alpha\theta_n)\widehat{e^3\dot{\sigma}_e}(t_L{}^n) = \frac{2S}{3}\hat{\dot{x}}^3(t_L)\sum_{j=1}^{n}(\omega_j - \omega_{j-1})\exp(\alpha\theta_j).$$

So that we can reduce our equation to:

$$\sum_{k=1}^{n}\exp(\alpha\theta_k)\left\{\alpha\sigma_i(t_L{}^k) - \alpha\sigma_i(t_L{}^k) - \alpha\sigma_i(t_L{}^k) - (\omega_k - \omega_{k-1})\left[h(x^3) + \frac{2S}{3}\hat{\dot{x}}^3(t_L{}^n)\right]\right\} = \exp(\alpha\theta_n)\dot{\sigma}_i(t_L{}^n). \tag{A6}$$

In a similar way, we can sum the lines L_1 to U_n, eliminate $\widehat{e^3\dot{\sigma}_e}(t_U{}^n)$ from (A5) and obtain:

$$\sum_{k=1}^{n}\exp(\alpha\theta_k)\left\{\alpha\sigma_i(t_L) - \alpha\sigma_i(t_U{}^{k-1}) - (\omega_k - \omega_{k-1})\left[h(x^3) + \frac{2S}{3}\hat{\dot{x}}^3(t_U{}^n)\right]\right\} = -\exp(\alpha\theta_{n+1})\dot{\sigma}_i(t_U{}^n)$$

(A6) and (A7) are the general solution of the problem.